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ON THE MONOTONICITY OF SOME FUNCTIONALS IN THE FAMILY OF UNIVALENT FUNCTIONS

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ABSTRACT

Let S denote the class of regular and univalent functions in $|z|<1$ with the normalization $f(0)=0$, $f'(0)=1$. Denote $d_f=\inf_{f\in S}\{|\alpha| \mid f(z)\neq\alpha, |z|<1\}$ and let $S(d)=\{f \mid f\in S, d_f=d, \frac{1}{4}\leqq d\leqq 1\}$. The analytic function $f(z)$ is univalent in $|z|<1$ if and only if

$$\log\frac{f(z)-f(\zeta)}{z-\zeta}=\sum_{m,n=0}^{\infty} d_{mn}z^m\zeta^n$$

converges in the bicylinder $|z|<1$, $|\zeta|<1$. Let $c_{mn}=\sqrt{mn}\,d_{mn}$ and $c_{nn}(d)=\mathrm{Max}_{f\in S(d)}\{\mathrm{Re}(c_{nn})\}$. The paper deals with the monotonicity of $c_{nn}(d)$ and related functionals.

1. Introduction

In dealing with extremum problems for univalent functions, it is relatively easy to characterize the extremum functions by functional-differential equations. A much harder task is the identification of the solution and the complete answer to the problem. In particular, one has often to select from a number of possible solutions to the functional-differential equation the actual extremum function. In this paper we discuss a question which is of interest in itself and for which a complete answer can be obtained by means of the variational method.

Let S be the family of regular normalized univalent functions in $|z|<1$:

$$f(z)=z+a_2z^2+\cdots. \tag{1}$$

For any $f\in S$ let

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$$d_f = \inf\{|\alpha| \mid f(z) \neq \alpha, |z| < 1\} \tag{2}$$

and denote by $S(d)$, $\frac{1}{4} \leqq d \leqq 1$ the sub-family of S for which $d_f = d$ (see [2]).

Let $\phi[f]$ be any functional defined for the class S and let $M(d)$ be the maximum value of $\phi[f]$ with respect to the subclass $S(d)$. Consider then a value d_0 for which $M(d_0)$ is stationary. Under any variation of the extremum function $f_0(z) \in S(d_0)$ which is admissible in S, the value $\phi[f_0]$ will be stationary; hence $f_0(z)$ will satisfy the functional-differential equation for the extremum of $\phi[f]$ within S. This shows the importance of the question whether this equation determines a solution in a unique way. Indeed, whenever this is so, $M(d)$ will be monotonically decreasing with d. If there are several solutions to the functional-differential equation, however, one may determine the largest d-value to which all of them belong and guarantee, at least, the monotonic decrease of $M(d)$ for all larger d.

We shall now introduce a particular functional $\phi[f]$ which has been already extensively studied within the family S.

The analytic function $f(z)$ is univalent in $|z| < 1$ if and only if the double series

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=0}^{\infty} d_{mn} z^m \zeta^n \tag{3}$$

converges in the bicylinder $|z| < 1$, $|\zeta| < 1$. It was shown by Grunsky [1] that a necessary and sufficient condition for the function $f(z)$ to be univalent is that the sharp inequalities

$$\mathrm{Re}\left\{\sum_{m,n=1}^{\infty} c_{mn}\lambda_m\lambda_n\right\} \leqq \sum_{n=1}^{\infty} |\lambda_n|^2 \tag{4}$$

hold for all choices of λ_m, where

$$c_{mn} = \sqrt{mn}\, d_{mn}. \tag{5}$$

With $f(z)$ we consider the odd univalent function

$$F(z) = \sqrt{f(z^2)} \tag{6}$$

belonging to S, and denote the Grunsky coefficients belonging to $F(z)$ by D_{mn} and C_{mn} respectively.

Because of the identity

$$\log \frac{F(z) - F(\zeta)}{z - \zeta} + \log \frac{F(z) + F(\zeta)}{z + \zeta} = \log \frac{f(z^2) - f(\zeta^2)}{z^2 - \zeta^2} \tag{7}$$

it follows that

$$C_{2m,2n} = c_{mn}. \tag{8}$$

We will consider the behavior of

$$(9)\quad \begin{aligned} &\text{(a)} \qquad c_{nn}(d) = \operatorname*{Max}_{f\in S(d)} \{\mathrm{Re}(c_{nn})\} \\ &\text{(b)} \qquad C_{nn}(d) = \operatorname*{Max}_{f\in S(d)} \{\mathrm{Re}(C_{nn})\} \end{aligned}$$

and prove the following

THEOREM. $C_{nn}(d) = 1$ *for all values of* d *in* $[\frac{1}{4}, (\frac{1}{4})^{1/n}]$ *and strictly monotone decreasing for* d *in* $[(\frac{1}{4})^{1/n}, 1)]$.

2. The case of a local maximum of $\mathrm{Re}(c_{nn})$

Because of the compactness of S there is, at least, one function for which $\mathrm{Max}_{f\in S}\{\mathrm{Re}(c_{nn})\}$ is attained. So let $w = f(z)$ be such a function and let w_0 be a value in the w-plane such that $f(z) \neq w_0$ for $|z| < 1$.

As shown in [3], there exist functions $f^*(z) \in S$ of the asymptotic form

$$f^*(z) = f(z) + \frac{a\rho^2 f(z)^2}{w_0^2(f(z) - w_0)} + o(\rho^2) \tag{10}$$

for $\rho > 0$ arbitrary small, $|a| = 1$, such that $o(\rho^2)$ can be estimated uniformly in each closed subdomain of $|z| < 1$.

A simple calculation gives that

$$\log \frac{f^*(z) - f^*(\zeta)}{z - \zeta} = \log \frac{f(z) - f(\zeta)}{z - \zeta} + \frac{a\rho^2}{w_0^2}\left(1 - \frac{1}{\left(1 - \frac{f(z)}{w_0}\right)\left(1 - \frac{f(\zeta)}{w_0}\right)}\right) + o(\rho^2). \tag{11}$$

Hence

$$\begin{aligned} \sum_{m,n=0}^{\infty} \delta d_{mn} z^m \zeta^n &= \frac{a\rho^2}{w_0^2}\left(1 - \frac{1}{\left(1 - \frac{f(z)}{w_0}\right)\left(1 - \frac{f(\zeta)}{w_0}\right)}\right) + o(\rho^2) \\ &= \frac{a\rho^2}{w_0^2} - \frac{a\rho^2}{w_0^2} \sum_{m,n=0}^{\infty} \gamma_m\left(\frac{1}{w_0}\right) \gamma_n\left(\frac{1}{w_0}\right) z^m \zeta^n + o(\rho^2), \end{aligned} \tag{12}$$

where

$$\frac{1}{1 - \frac{f(z)}{w_0}} = \sum_{m=0}^{\infty} \gamma_m\left(\frac{1}{w_0}\right) z^m \tag{13}$$

and $\gamma_m(x)$ is a polynomial in x of degree m with $\gamma_m(0)=0$, $m \geqq 1$, starting with x^m (δ stands, as usual, for variation).

Thus

$$\delta d_{mn} = -\frac{a\rho^2}{w_0^2}\gamma_m\left(\frac{1}{w_0}\right)\gamma_n\left(\frac{1}{w_0}\right)+o(\rho^2), \qquad m,n \geqq 1. \tag{14}$$

Because of the extremal property of $f(z)$ it follows that

$$\operatorname{Re}\left\{-\frac{a\rho^2}{w^2}\gamma_n\left(\frac{1}{w}\right)^2+o(\rho^2)\right\} \leqq 0, \tag{15}$$

for any value w not in the image of $|z|<1$, and admissible variations (10).

Applying the fundamental lemma of the calculus of variations for univalent functions [3] we get that an extremal function maps $|z|<1$ onto the whole w-plane slit along analytic arcs $w(t)$, each of which satisfies the differential equation

$$\left(\frac{dw}{dt}\right)^2\frac{1}{w^2}\gamma_n\left(\frac{1}{w}\right)^2 \leqq 0 \tag{16}$$

for any real curve parameter t.

If we parametrize the boundary arcs in the form

$$w(\phi)=f(e^{i\phi}),\quad \frac{dw}{d\phi}=ie^{i\phi}f'(e^{i\phi}); \qquad 0 \leqq \phi < 2\pi, \tag{17}$$

we get from (16) that

$$\frac{z^2f'(z)^2}{f(z)^2}\gamma_n\left(\frac{1}{f(z)}\right)^2 \geqq 0; \qquad |z|=1. \tag{18}$$

Hence

$$\frac{zf'(z)}{f(z)}\gamma_n\left(\frac{1}{f(z)}\right)=\text{real} \tag{19}$$

on $|z|=1$.

The left-hand side of (19) is regular in $|z|<1$ except for a pole of order n at $z=0$, and real on $|z|=1$.

By the Schwarz reflection principle it can be continued for $|z|>1$.

Thus $f(z)$ satisfies in the whole z-plane the differential equation

$$\frac{zf'(z)}{f(z)}\gamma_n\left(\frac{1}{f(z)}\right)=\frac{1}{z^n}+\cdots+z^n, \tag{20}$$

where the expression on the right of (20) is a rational function real on $|z|=1$.

We continue along lines similar to those in [4].

From the development

$$\log(1 - tf(z)) = -\sum_{n=1}^{\infty} g_n(t) z^n, \tag{21}$$

where $g(t)$ is a polynomial in t of degree n, we get by differentiation with respect to t that

$$\frac{f(z)}{1 - tf(z)} = \sum_{n=1}^{\infty} g_n'(t) z^n. \tag{22}$$

Hence, from (13),

$$t g_n'(t) = \gamma_n(t), \qquad n \geqq 1. \tag{23}$$

Comparing the coefficients of ζ^n on both ends of

$$\begin{aligned} \sum_{m,n=0}^{\infty} d_{mn} z^m \zeta^n &= \log \frac{f(z)}{z} + \log\left(1 - \frac{f(\zeta)}{f(z)}\right) - \log\left(1 - \frac{\zeta}{z}\right) \\ &= \log \frac{f(z)}{z} - \sum_{n=1}^{\infty} g_n\left(\frac{1}{f(z)}\right)\zeta^n + \sum_{n=1}^{\infty} \frac{\zeta^n}{n z^n} \end{aligned} \tag{24}$$

we get that

$$\sum_{m=1}^{\infty} d_{mn} z^m = -g_n\left(\frac{1}{f(z)}\right) + \frac{1}{n z^n}. \tag{25}$$

From (25), by differentiation,

$$\frac{f'(z)}{f(z)^2} g_n'\left(\frac{1}{f(z)}\right) = \frac{1}{z^{n+1}} + \sum_{m=1}^{\infty} m d_{mn} z^{m-1}, \tag{26}$$

and with (23)

$$\frac{zf'(z)}{f(z)} \gamma_n\left(\frac{1}{f(z)}\right) = \frac{1}{z^n} + \sum_{m=1}^{\infty} m d_{mn} z^m. \tag{27}$$

Because of (20), it now follows that

$$m d_{mn} = \begin{cases} 0 & m \neq n, \\ 1 & m = n. \end{cases} \tag{28}$$

Hence

$$\operatorname*{Max}_{f \in S} \{\mathrm{Re}(c_{nn})\} = 1 \tag{29}$$

and an extremal function satisfies

$$\frac{zf'(z)}{f(z)}\gamma_n\left(\frac{1}{f(z)}\right)=\frac{1}{z^n}+z^n. \tag{30}$$

Note that (29) and (30) have been proved on the assumption that $\mathrm{Re}(c_{nn})$ for $f(z)$ is *not* less than $\mathrm{Re}(c_{nn})$ for the functions $f^*(z)$ "near" $f(z)$, i.e. when $\mathrm{Re}(c_{nn})$ has a local maximum.

This remark is important in the sequel.

Integrating (30) we get that

$$\int_\infty^f \frac{1}{f}\gamma_n\left(\frac{1}{f}\right)df=\int_{z_0}^z\left(\frac{1}{z^{n+1}}+z^{n-1}\right)dz, \tag{31}$$

where $|z_0|=1$ and $f(z_0)=\infty$, and end up with

$$P\left(\frac{1}{f(z)}\right)\equiv\frac{1}{f(z)^n}+\cdots+\frac{A_{n-1}}{f(z)}=\frac{1}{z^n}-z^n-C \tag{32}$$

where C is a pure imaginary constant. Because $1/z^n-z^n$ is also pure imaginary on $|z|=1$ there is at least one value z_1, $|z_1|=1$, for which

$$\left|\frac{1}{z_1^n}-z_1^n-C\right|\geqq 2. \tag{33}$$

Consider now the algebraic equation

$$P\left(\frac{1}{w}\right)=\frac{1}{z_1^n}-z_1^n-C, \tag{32'}$$

which has n roots w_k, $k=0,1,\cdots,n-1$. These have preimages z_k of the form

$$\text{either } z_1\exp\left(\frac{2\pi i}{n}k\right)\quad\text{or}\quad 1/z_1\exp\left(\frac{2\pi i}{n}\left(k+\frac{1}{2}\right)\right).$$

At any rate, all roots, w_k, lie on the boundary of the image domain. Hence, in view of (32′) and (33) we have

$$\frac{1}{\left(\operatorname{Min}_{1\leq k\leq n}|w_k|\right)^n}\geqq\frac{1}{\prod_{k=1}^n|w_k|}\geqq 2, \tag{34}$$

or

$$\operatorname{Min}_{1\leq k\leq n}|w_k|\leqq\left(\frac{1}{4}\right)^{1/2n}. \tag{35}$$

Thus, we have proved that an extremal function must belong to $S(d)$, where

$$\frac{1}{4} \leqq d \leqq \left(\frac{1}{4}\right)^{1/2n}. \tag{36}$$

We are to show now that for *every* d satisfying (36) there is an extremal function belonging to $S(d)$ for which $c_{nn} = 1$.

We claim that the functions

$$f(z,t) = \frac{e^t}{1/i^n} k[(-k^{-1}(e^{-2nt}k(z^{2n})))^{1/2n}], \qquad 0 \leqq t \leqq \infty, \tag{37}$$

where k is the Koebe function $k(z) = z/(1-z)^2$, have the desired property.

From (37)

$$f(z,0) = \frac{z}{(1-i^{1/n}z)^2}, \qquad f(z,\infty) = \frac{z}{(1-z^{2n})^{1/n}}. \tag{38}$$

By direct calculation, we get that for these two functions $c_{nn} = 1$. So $(1/i^{1/n})k(i^{1/n}z)$ satisfies (32) with a corresponding $P(1/f)$.

Let us write this equation in the form

$$P\left(\frac{i^{1/n}}{e^t k(i^{1/n}\zeta)}, t\right) = e^{-nt}\left(\frac{1}{\zeta^n} - \zeta^n - C\right), \qquad |\zeta| < 1, \tag{39}$$

where the coefficients of the polynomial P on the left-side of (39) are now dependent on t.

Writing in (37)

$$\zeta = (k^{-1}(e^{-2nt}k(z^{2n})))^{1/2n}, \qquad \eta = e^{-2nt}k(z^{2n}) \tag{40}$$

we get from (39) that

$$P\left(\frac{1}{f(z,t)}, t\right) = e^{-nt}\left(\frac{1}{\sqrt{k^{-1}(\eta)}} - \sqrt{k^{-1}(\eta)} - C\right) = e^{-nt}\left(\frac{1}{\sqrt{\eta}} - C\right) \tag{41}$$

$$= \frac{1}{z^n} - z^n - C_1,$$

where $C_1 = Ce^{-nt}$.

Hence, $f(z,t)$ will satisfy (30) with the corresponding γ_n. But then it follows from (27) that c_{nn} for this function is equal to 1.

We calculate now the nearest point to the origin in the image of $|z| < 1$ by $f(z,t)$.

η maps $|z| < 1$ onto the whole plane slit along $-\infty \leqq \lambda \leqq -\frac{1}{4}e^{-2nt}$. ζ maps this plane onto $|\zeta|<1$ slit along $2n$ radial slits, one of them being $-1 \leqq \eta \leqq -\tau$, $0<\tau<1$.

An easy calculation shows that the nearest point in the image of $|z|<1$ by $(1/i^{1/n})k(i^{1/n}\zeta)$ is the image of $\tau e^{(2n-1)\pi i/2n}$. The distance of this image from the origin is $\tau/(1+2\tau\cos(\pi/2n)+\tau^2)$.

Thus $f(z,t) \in S(d(t))$ where

$$d(t)=\frac{e^t\tau}{1+2\tau\cos\dfrac{\pi}{2n}+\tau^2}. \tag{42}$$

As τ is a continuous function of t, so is $d(t)$. From (38) we have that $d(0)=\frac{1}{4}$ and $d(\infty)=(\frac{1}{4})^{1/2n}$. Hence $d(t)$ takes every value in $[\frac{1}{4},(\frac{1}{4})^{1/2n}]$.

Because of (8) the result proved in this paragraph holds for $C_{2n,2n}(d)$.

3. The case of a local maximum of $\mathrm{Re}(C_{nn})$

Let $F(z)$ be an odd univalent function for which $\mathrm{Re}(C_{nn})$ attains a maximum, and let $f(z)$ be connected with $F(z)$ by (6).

Because of (8) we need to consider only the case where n is odd.

Performing the variation of $f(z)$ given by (10) we get that

$$\begin{aligned} F^*(z) &= \sqrt{f^*(z^2)} = \sqrt{f(z^2)}\left(1+\frac{a\rho^2 f(z^2)}{2w_0^2(f(z^2)-w_0)}\right)+o(\rho^2) \\ &= F(z)+\frac{a\rho^2}{2w_0^2}\frac{F(z)^3}{F(z)^2-w_0}+o(\rho^2), \end{aligned} \tag{43}$$

where w_0 is any fixed value not in the image of $|z|<1$ by $f(z)$.

An elementary calculation gives

$$\begin{aligned} \log\frac{F^*(z)-F^*(\zeta)}{z-\zeta} &= \log\frac{F(z)-F(\zeta)}{z-\zeta} \\ &+\log\left[1+\frac{a\rho^2}{2w_0^2}\left(1-\frac{w_0^2+w_0F(z)F(\zeta)}{(F(z)^2-w_0)(F(\zeta)^2-w_0)}\right)\right]+o(\rho^2) \end{aligned} \tag{44}$$

and hence

$$\delta\log\frac{F(z)-F(\zeta)}{z-\zeta}=\frac{a\rho^2}{2w_0^2}\left(1-\frac{w_0^2+w_0F(z)F(\zeta)}{(F(z)^2-w_0)(F(\zeta)^2-w_0)}\right)+o(\rho^2). \tag{45}$$

Since $F(z)^2$ is an even function

$$\sum_{m,n=1}^{\infty}\delta D_{2m+1,2n+1}z^{2m+1}\zeta^{2n+1}=-\frac{a\rho^2F(z)F(\zeta)}{2w_0(F(z)^2-w_0)(F(\zeta)^2-w_0)}+o(\rho^2). \tag{46}$$

From the development

$$\frac{F(z)}{1-xF(z)^2}=\sum_{m=0}^{\infty}\Gamma_m(x)z^{2m+1} \tag{47}$$

where $\Gamma_m(x)$ is a polynomial in x of degree m, we find that

$$\sum_{m,n=0}^{\infty}\delta D_{2m+1,2n+1}z^{2m+1}\zeta^{2n+1}=-\frac{a\rho^2}{2w_0^3}\sum_{m,n=0}^{\infty}\Gamma_m\left(\frac{1}{w_0}\right)\Gamma_n\left(\frac{1}{w_0}\right)z^{2m+1}\zeta^{2n+1}+o(\rho^2). \tag{48}$$

Hence

$$\delta D_{2n+1,2n+1}=-\frac{a\rho^2}{2w_0^3}\Gamma_n\left(\frac{1}{w_0}\right)^2+o(\rho^2). \tag{49}$$

Because $\{\operatorname{Re} D_{2n+1,2n+1}\}$ is maximal it follows that

$$\operatorname{Re}\left\{-\frac{a\rho^2}{w^3}\Gamma_n\left(\frac{1}{w}\right)^2+o(\rho^2)\right\}\leqq 0, \tag{50}$$

for any value w not in the image, and admissible variations (10).

Applying again the fundamental lemma of the calculus of variations for univalent functions we find that the extremal function $w=f(z)$ maps $|z|<1$ onto the whole w-plane slit along analytic arcs each of which satisfies the differential equation

$$\left(\frac{dw}{dt}\right)^2\frac{1}{w^3}\Gamma_n\left(\frac{1}{w}\right)^2+1=0, \tag{51}$$

for a proper choice of the real curve parameter t.

Choosing

$$w(\phi)=f(e^{i\phi}),\qquad 0\leqq\phi<2\pi, \tag{52}$$

we get from (51) that

$$\frac{z^2f'(z)^2}{f(z)^3}\Gamma_n\left(\frac{1}{f(z)}\right)^2\geqq 0\qquad \text{on } |z|=1. \tag{53}$$

The expression on the left of (53) is regular in $|z|<1$ except for a pole of order $2n+1$ at the origin and real on $|z|=1$.

Hence, by the Schwarz reflection principle

$$\frac{z^2f'(z)^2}{f(z)^3}\Gamma_n\left(\frac{1}{f(z)}\right)^2=\frac{1}{z^{2n+1}}+\frac{A_{2n}}{z^{2n}}+\cdots+A_0+\cdots+\bar{A}_{2n}z^{2n}+z^{2n+1}, \tag{54}$$

with A_0 real, holds for all z.

On the other hand we have that

$$\log \frac{z+\zeta}{z-\zeta} + \log \frac{F(z)-F(\zeta)}{F(z)+F(\zeta)} = 2 \sum_{m,n=0}^{\infty} D_{2m+1,2n+1} z^{2m+1} \zeta^{2n+1}. \tag{55}$$

From (55) and the development

$$\log \frac{1-tF(\zeta)}{1+tF(\zeta)} = -2 \sum_{n=0}^{\infty} G_{2n+1}(t) \zeta^{2n+1}, \tag{56}$$

where $G_{2n+1}(t)$ is a polynomial in t of degree $2n+1$, we get that

$$G_{2n+1}\left(\frac{1}{F(z)}\right) = \frac{1}{(2n+1)z^{2n+1}} - \sum_{m=0}^{\infty} D_{2m+1,2n+1} z^{2m+1}. \tag{57}$$

Defferentiation of (56) with respect to t gives

$$\frac{F(\zeta)}{1-t^2F(\zeta)^2} = \sum_{n=0}^{\infty} G'_{2n+1}(t) \zeta^{2n+1}. \tag{58}$$

Combining (58) with (47) yields

$$\Gamma_n(t^2) = G'_{2n+1}(t). \tag{59}$$

From (57) it follows that

$$G'_{2n+1}\left(\frac{1}{F(z)}\right) \frac{F'(z)}{F(z)^2} = \frac{1}{z^{2n+2}} + \sum_{m=0}^{\infty} (2m+1) D_{2m+1,2n+1} z^{2m}. \tag{60}$$

As

$$F'(z) = \frac{zf'(z^2)}{\sqrt{f(z^2)}}, \tag{61}$$

it follows from (59) and (60) that

$$\frac{zf'(z^2)}{f(z^2)^{3/2}} \Gamma_n\left(\frac{1}{f(z^2)}\right) = \frac{1}{z^{2n+2}} + \sum_{m=0}^{\infty} (2m+1) D_{2m+1,2n+1} z^{2m}. \tag{62}$$

Substituting z for z^2 and multiplying by $\sqrt{z}$, we end up with

$$\frac{zf'(z)}{f(z)^{3/2}} \Gamma_n\left(\frac{1}{f(z)}\right) = \frac{1}{z^{n+1/2}} + \sum_{m=0}^{\infty} (2m+1) D_{2m+1,2n+1} z^{m+1/2}. \tag{63}$$

Because of (53) the right-hand side of (63) must be real on $|z|=1$. Hence

$$(2m+1) D_{2m+1,2n+1} = \begin{cases} 0 & m \neq n, \\ 1 & m = n. \end{cases} \tag{64}$$

So we have proved that $\max\{\mathrm{Re}(C_{2n+1,2n+1})\} = 1$ and an extremal function satisfies the differential equation

$$\frac{zf'(z)}{f(z)^{3/2}}\Gamma_n\left(\frac{1}{f(z)}\right) = \frac{1}{z^{n+\frac{1}{2}}} + z^{n+\frac{1}{2}}. \tag{65}$$

Integration of (65) gives

$$P\left(\frac{1}{f(z)}\right) \equiv \frac{1}{f(z)^{\frac{1}{2}}}\left(\frac{1}{f(z)^n} + \frac{A_1}{f(z)^{n-1}} + \cdots + A_n\right) = \frac{1}{z^{n+\frac{1}{2}}} - z^{n+\frac{1}{2}}; \tag{66}$$

note that the constant of integration is zero.

Squaring both sides of (66) gives that

$$\frac{1}{f(z)^{2n+1}} + \cdots + \frac{A_n^2}{f(z)} = \frac{(1 - z^{2n+1})^2}{z^{2n+1}}. \tag{67}$$

For any value of z on $|z| = 1$, the algebraic equation (67) has $(2n+1)$ solutions. Denote by $w_1, \cdots, w_{2n+1}$ the values of $f(z)$ in (67) for $z^{2n+1} = -1$. We have

$$\prod_{i=1}^{2n+1} |w_i| = \frac{1}{4}, \tag{68}$$

and hence

$$\min_{1 \leqq i \leqq 2n+1} \{|w_i|\} \leqq \left(\frac{1}{4}\right)^{1/(2n+1)}. \tag{69}$$

This last inequality implies that the extremal function $f(z)$ belongs to $S(d)$ where

$$\frac{1}{4} \leqq d \leqq \left(\frac{1}{4}\right)^{1/(2n+1)}. \tag{70}$$

Our aim is to show that for *every* d satisfying (70) there is an extremal function for which $C_{2n+1,2n+1} = 1$.

We first note that any function $f(z)$ satisfying (66) will satisfy (65) with the corresponding Γ_n. Together with (63) for this Γ_n will give (64). Hence $f(z)$ will be an extremal function for which $C_{2n+1,2n+1} = 1$.

A direct calculation shows that $C_{2n+1,2n+1} = 1$ for the Koebe function. Therefore

$$P\left(\frac{1}{k(z)}\right) = \frac{1}{z^{n+\frac{1}{2}}} - z^{n+\frac{1}{2}}. \tag{71}$$

Taking in (71) $e^t k(z)$ instead of $k(z)$, $0 \leqq t < \infty$ we get

$$P\left(\frac{1}{e^t k(z)}, t\right) = e^{-(n+\frac{1}{2})t}\left(\frac{1}{z^{n+\frac{1}{2}}} - z^{n+\frac{1}{2}}\right) \tag{72}$$

where the coefficients of P in (72) depend now on t.

We claim that the functions

$$f(z,t) = e^t k[(k^{-1}(e^{-(2n+1)t}k(z^{2n+1})))^{1/(2n+1)}], \qquad |z| < 1, \tag{73}$$

t fixed, $0 \leqq t < \infty$, belong to S and satisfy (66), and so are extremal functions for which $C_{2n+1,2n+1} = 1$.

Indeed, putting

$$\zeta = (k^{-1}(\eta))^{1/(2n+1)}, \qquad \eta = e^{-(2n+1)t}k(z^{2n+1}) \tag{74}$$

we get from (72) that

$$P\left(\frac{1}{f(z,t)}, t\right) = e^{-(n+\frac{1}{2})t}\left(\frac{1}{\sqrt{k^{-1}(\eta)}} - \sqrt{k^{-1}(\eta)}\right) \tag{75}$$

$$= e^{-(n+\frac{1}{2})t}\frac{1}{\sqrt{\eta}} = \frac{1}{z^{n+\frac{1}{2}}} - z^{n+\frac{1}{2}}.$$

By (74) the disk $|z| < 1$ is mapped onto $|\zeta| < 1$ with $(2n+1)$ radial slits of equal length, one of them from -1 to $-\tau$, $0 < \tau < 1$, where

$$k(-\tau^{1/(2n+1)}) = -\frac{e^{(2n+1)t}}{4}. \tag{76}$$

From

$$|k(\zeta)| \geqq \frac{|\zeta|}{(1+|\zeta|)^2}, \tag{77}$$

it follows that the nearest boundary point to the origin in the image of $|z| < 1$ by $k(\zeta)$ is $k(-\tau)$.

Hence $f(z) \in S(d(t))$, where

$$d(t) = -e^t k[(k^{-1}(-\tfrac{1}{4}e^{-(2n+1)t}))^{1/(2n+1)}]. \tag{78}$$

$d(t)$ is a continuous function of t for $0 \leqq t \leqq \infty$. As $d(0) = \frac{1}{4}$ and $d(\infty) = (\frac{1}{4})^{1/(2n+1)}$, $d(t)$ takes every value in $[\frac{1}{4}, (\frac{1}{4})^{1/(2n+1)}]$.

The results obtained in the last two paragraphs together with the remark about the monotonicity in §1 complete the proof of the theorem.

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